

# Approximating Symmetric Linear Systems with recursive partial-Hankel Series Expansion

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**Abstract.** This paper presents an idea of transforming and approximating a symmetric linear system into a partial-Hankel series system. The motivation behind the transformation is to try to enable fast algorithms - typically  $T(n) = O(n \log n)$  - for solving Hankel and Toeplitz linear systems - to be used to solve general symmetric linear systems, e.g. found in regression problems. Symmetric linear systems typically have  $T(n) = O(n^3)$  (with Conjugate Gradient method) or at best  $T(n) = O(n^{2+\alpha})$ ;  $\alpha \in (0, 1)$  (with Strassen's method).

**Keywords:** Hankel, Toeplitz, Matrix Series Expansion

## 1 Introduction

**Definition 1 (Partial-Hankel).** A partial-Hankel matrix ( $n \times n$ ) is a symmetric matrix with a Hankel matrix ( $k \times k$ ;  $k \leq n$ ) with a boundary of 0's of size  $\frac{n-k}{2}$  centered around the Hankel matrix. If  $k = n$  a partial-Hankel matrix is an ordinary Hankel matrix.

**Definition 2 (Peel Matrix).** An peel matrix ( $n \times n$ ) is a matrix with the outermost non-zero region set to zero. This is performed by the peel() function, e.g.

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{pmatrix} \quad (1)$$

$$\text{peel}(B) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & 0 \\ 0 & b_{23} & b_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

**Theorem 1.** A symmetric matrix  $B$  ( $n \times n$ ) is equivalent to a series of  $\frac{n}{2}$  partial-Hankel matrices.

*Proof.* (sketch)

Let  $B$  be a  $4 \times 4$  symmetric matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{22} & b_{23} & b_{24} \\ b_{13} & b_{23} & b_{33} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{pmatrix} \quad (3)$$

Let  $C$  be a  $4 \times 4$  Hankel matrix generated by the exterior rows and columns of  $B$ .

$$C = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{12} & b_{13} & b_{14} & b_{24} \\ b_{13} & b_{14} & b_{24} & b_{34} \\ b_{14} & b_{24} & b_{34} & b_{44} \end{pmatrix} \quad (4)$$

Let  $D = \text{peel}(C)$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & 0 \\ 0 & b_{14} & b_{24} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (5)$$

and  $E = \text{peel}(B)$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & 0 \\ 0 & b_{23} & b_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (6)$$

Then

$$\text{partial-Hankel-series}(B) = C - \text{peel}(C) + \text{peel}(B) = C - D + E \quad (7)$$

**In the general case with  $n \times n$  matrix  $B$  :**  $E$  is symmetric, so we get a recursive peeling property holds and we get  $B$  is a sum of partial-Hankel matrices, each of the matrices in the sum become increasingly sparse (but dense and Hankel-type around the middle).

#### general recursive case

$$\text{partialHankelSeries}(B) = \text{exteriorHankelize}(B) - \quad (8)$$

$$\text{peel}(\text{exteriorHankelize}(B)) + \quad (9)$$

$$\text{partialHankelSeries}(\text{peel}(B)) \quad (10)$$

*Conjecture 1 (Approximation).*  $B$  may be approximated by adding together  $t \leq n$  partial-Hankel matrices. If  $t = n$  the sum is equivalent to  $B$ .

## 2 Open Questions

1. Is this a valid and well-known transformation?
2. How to determine the approximation accuracy as a function of  $t$ ?
3. Can this approximation be used sequentially to efficiently solve symmetric linear systems?
4. Can this approximation be used in parallel to efficiently solve symmetric linear systems?